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ANALYSIS OF THE BASIC MATHEMATICAL METHODS USED FOR DESCRIPTION OF THE MOTION OF AN OBJECT ON A PLANE AND IN THE SPACE

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Introduction

The procedure for rotating an object around an axis is a fundamental task in computer graphics. It is essential for constructing axonometric projections [1], as well as for representing and processing digital images [2]. Furthermore, this method is indispensable for describing the orientation of a rotating solid body around a fixed point relative to an absolute coordinate system [3].

This study analyzes the primary mathematical frameworks used to describe the kinematics of such objects. Particular attention is paid to the method of quaternions (hypercomplex numbers), which enables efficient modeling of rotations in three-dimensional space. Unlike conventional Euler angles, the application of quaternion algebra (based on the quaternion group Q_8) avoids mathematical singularities known as "Gimbal Lock" [4]. This approach ensures superior computational stability and speed during motion interpolation in real-time systems [5].

Objective of research

The aim of the article is to conduct a comparative analysis of different mathematical tools related to linear algebra used to solve a wide range of applied problems. In particular, the features of using Euler angles, rotation matrices, and quaternions are considered in the context of autonomous vehicles [6], missile guidance [7], and robotic manipulators [8]. An important aspect is the methodology of teaching these linear algebra concepts to future IT and aviation specialists, emphasizing the use of modern information technologies [9, 10].

Main Body

Let's consider the basic mathematical constructs used to describe the motion of various mechanisms, in particular, vehicles (piloted and autonomous) and robots.

Consider the rotation of an object around an axis and the rotation of a rigid body relative to an absolute coordinate system. If unit radii are used, then it is possible to make the transition from polar coordinates to defining trigonometric functions as directional relations (the three components of a unit vector are called its direction cosines).

Euler angles

The Euler angles are three angles introduced by Leonhard Euler to describe the orientation of a rigid body with respect to a fixed coordinate system [11]. They can also be used to describe the orientation of a common basis in three-dimensional linear algebra or to represent the orientation of a moving frame of reference in physics. In aviation, the sequence of roll, pitch, and yaw is used to describe aircraft movement in the O*xyz* system [12].

Euler angles define three rotations of the system that allow any position of the system to be brought to its current position. Let denote the initial system as (*x, y, z*), and the final system as (x', y', z'). The intersection of the coordinate planes is called the line of nodes. The angle between the axis and the line of nodes is the precession angle. The angle between the axes and is the nutation angle. The angle between the line of nodes and the axis is the proper rotation angle.

Rotation matrix

In practice, the general triple rotation matrix (rotation matrix) is very often used. Triple rotation matrix is formed by multiplying three rotation matrices: rotation around the *X*-axis, rotation around the *Y*-axis, and rotation around the *Z*-axis.

Let consider description of the movement of aircraft in Cartesian coordinate system O*xyz*, connected with this plane. In aviation, the so-called third system of Euler's angles is used to describe the movement of aircraft in Cartesian coordinate system, which corresponds to the following sequence of rotations: a rotation around the roll axis, which coincides with the longitudinal axis of the aircraft; a rotation around the pitch axis; and a rotation around the yaw axis.

In practice, the general triple rotation matrix (rotation matrix) is very often used. Triple rotation matrix is formed by multiplying three rotation matrices: rotation around the roll axis, rotation around the pitch axis, and rotation around the yaw axis. Let consider rotation by an angle θin Cartesian coordinate system O*xyz*.

Rotation around X-axis: ,

Rotation around Y-axis

$$\begin{cases} \acute{x} = xcos(\theta) + zsin(\theta) \\ \acute{y} = y \\ \acute{z} = -xsin(\theta) + zcos(\theta) \end{cases},$$

Rotation around Z-axis:

$$\begin{cases} \acute{x} = xcos(\theta) - ysin(\theta) \\ \acute{y} = xsin(\theta) + ycos(\theta) \\ \acute{z} = z \end{cases}$$

consider this transformation in vector form Then rotation matrix will be:

around X -axis

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos\,(\theta) & -sin\,(\theta) \\ 0 & sin\,(\theta) & cos\,(\theta) \end{pmatrix},$$

around Y-axis

$$R_y(\theta) = \begin{pmatrix} cos(\theta) & 0 & sin(\theta) \\ 0 & 1 & 0 \\ -sin(\theta) & 0 & cos(\theta) \end{pmatrix},$$

around *Z*-axis

$$R_z(\theta) = \begin{pmatrix} cos\,(\theta) & -sin\,(\theta) & 0 \\ sin\,(\theta) & cos\,(\theta) & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

Let column-vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ be given.

Then rotation around X-axis is described by the formula:

$$\begin{bmatrix} \acute{x} \\ \acute{y} \\ \acute{z} \end{bmatrix} = R_x(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

rotation around *Y*-axis is described by the formula:

$$\begin{bmatrix} \acute{x} \\ \acute{y} \\ \acute{z} \end{bmatrix} = R_y(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

rotation around *Z* -axis is described by the formula:

$$\begin{bmatrix} \acute{x} \\ \acute{y} \\ \acute{z} \end{bmatrix} = R_z(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

These constructs are fundamental for 3D engines such as OpenGL, Unity, and Blender. The integration of such mathematical models into the training of IT specialists is crucial for developing professional skills in computer modeling [13]. Modern educational approaches often utilize cloud-based tools like Google Workspace to enhance the mathematical training of specialists in the field of Information Technology [14]. However, despite their clarity, matrix methods can be computationally redundant compared to alternative frameworks [15].

Quaternions

In group theory, the quaternion group Q_8 (sometimes just denoted by Q) is a non-abelian group of order eight, isomorphic to the eight-element subset of the quaternions under multiplication. It is given by the group presentation Q_8. These relations, discovered by W. R. Hamilton, also generate the quaternions as an algebra over the real numbers.

Another representation of Q_8 is given in the multiplication table.

Table 1. Quaternion group multiplication table (simplified form)

	1	*i*	*j*	*k*
1	1	*i*	*j*	*k*
i	*i*	-1	*k*	-*j*
j	*j*	-*k*	-1	*i*
k	*k*	-*j*	-*i*	-1

The simplified multiplication table (Table 1) illustrates the core relations between the fundamental imaginary units. To provide a rigorous description of the group Q_8, it is necessary to consider the full set of eight elements, including the negative units. This comprehensive interaction is detailed in the Cayley table (table 2).

Table 2. Cayley table for quaternions

	1	-1	*i*	-*i*	*j*	-*j*	*k*	-*k*
1	1	-1	*i*	-*i*	*j*	-*j*	*k*	-*k*
-1	-1	1	-*i*	*i*	-*j*	*j*	-*k*	*k*
i	*i*	-*i*	1	-1	*k*	-*k*	-*j*	*j*
-*i*	-*i*	*i*	-1	1	-*k*	*k*	*j*	-*j*
j	*j*	-*j*	-*k*	*k*	1	-1	*i*	-*i*
-*j*	-*j*	*j*	*k*	-*k*	-1	1	-*i*	*i*
k	*k*	-*k*	*j*	-*j*	-*i*	*i*	1	-1
-*k*	-*k*	*k*	-*j*	*j*	*i*	-*i*	-1	1

Table 2 fully defines the structure of the quaternion group Q_8. From a kinematic perspective, this algebraic structure is exceptionally useful: while Table 1 shows the "direction" of rotation between axes, the full Cayley table provides a complete mathematical framework for calculating complex spatial transformations without losing orientation data. This algebraic consistency allows quaternions to bypass the limitations of Euler angles and ensures high efficiency in processing gyroscope measurements [16].

Conclusion

The problems of mathematical support for controlling various mechanisms and vehicles have been relevant for many years. Currently, there is an increased interest in the creation of control systems for unmanned vehicles (aerial, ground, underwater, etc). Therefore, an important factor in the success of the process is good training of both developers and operators.

The article provides an analysis of the primary mathematical methods used for modeling the motion and orientation of objects in space. Based on the conducted research, the following conclusions can be drawn:

1. Matrix methods and Euler angles remain fundamental due to their visual clarity; however, they possess significant limitations: high computational redundancy and the risk of mathematical singularities (Gimbal Lock), which is critical for navigation systems.
2. The quaternion framework, based on the non-commutative algebra of the Q_8group, proves to be the most effective solution for describing rotations, ensuring stability in aircraft trajectory formation [17].
3. Integration of Tracking Algorithms: For the practical implementation of autonomous functions, such as target following, mathematical models of motion must be combined with robust computer vision. As shown in [19], the use of specialized object tracking algorithms based on video images is essential for the realization of autonomous target-following functions for UAVs.

This synergy between spatial kinematics and real-time visual tracking provides the necessary reliability for mission execution.

4. Educational Perspective: The effective training of future IT and aviation specialists requires a deep integration of linear algebra with modern software tools and information technologies [9, 13, 14].

In conclusion, for the development of high-reliability software in avionics and 3D modeling, the use of quaternion models is prioritized as the most optimal mathematical foundation.

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АНАЛІЗ ОСНОВНИХ МАТЕМАТИЧНИХ МЕТОДІВ, ЩО ВИКОРИСТОВУЮТЬСЯ ДЛЯ ОПИСУ РУХУ ОБ'ЄКТА НА ПЛОЩИНІ ТА У ПРОСТОРІ

У статті наведено порівняльний аналіз ключових математичних інструментів лінійної алгебри, що використовуються для моделювання орієнтації та руху об'єктів. У дослідженні розглянуто конкретне застосування кутів Ейлера, матриць обертання та кватерніонів у контексті автономного керування транспортними засобами. Розглянуто питання про переваги і недоліки кожного з цих підходів. Зокрема, кожен з цих підходів має свої переваги і недоліки. Розглянуто питання про переваги кожного з цих підходів. Певну увагу слід приділяти методології викладання цих понять майбутнім IT-фахівцям та авіаційним інженерам з використанням сучасних інформаційних технологій.

Ключові слова*: лінійна алгебра, кути Ейлера, кватерніони, матриці повороту, автономні системи.*

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ANALYSIS OF THE BASIC MATHEMATICAL METHODS USED FOR DESCRIPTION OF THE MOTION OF AN OBJECT ON A PLANE AND IN THE SPACE

The article provides a comparative analysis of key mathematical tools in linear algebra used for modeling the orientation and motion of objects. The study examines the specific applications of Euler angles, rotation matrices, and quaternions within the context of autonomous vehicle control. The advantages of each of these approaches are discussed. Some attention should be paid to the methodology of teaching the seconcepts to future IT specialists and aviation engineers using modern information technologies.

Keywords*: linear algebra, Euler angles, quaternions, rotation matrices, autonomous systems.*

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